



TITLE:

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# BSPFA Combined with One Measurable Cardinal

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## Abstract

We consider consequences of BSPFA (Bounded Semi-Proper Forcing Axiom) combined with an existence of a measurable cardinal. The large cardinal assures existences of relevant semiproper preorders via Chang's Conjecture-type arguments.

## Introduction

In [T], a new combinatorial principle  $\theta_{AC}$  is introduced. We recall its definition.

**Definition.** ([T])  $\theta_{AC}$  holds, if for every one-to-one list  $\mathbf{r} = \langle r_i \mid i < \omega_1 \rangle$  in  ${}^\omega 2$  and every  $S \subseteq \omega_1$ , there exist ordinals  $\gamma > \beta > \alpha \geq \omega_1$  and an increasing continuous decomposition  $\gamma = \bigcup \{N_\nu \mid \nu < \omega_1\}$  of the ordinal  $\gamma$  into countable sets such that for all  $\nu < \omega_1$ ,  $N_\nu \cap \omega_1 \in S$  if and only if the following holds, where  $i = \text{o.t.}(N_\nu \cap \alpha)$ ,  $j = \text{o.t.}(N_\nu \cap \beta)$  and  $k = \text{o.t.}(N_\nu)$ ,

$$\Delta(r_i, r_j) = \text{Max} \{ \Delta(r_i, r_j), \Delta(r_i, r_k), \Delta(r_j, r_k) \}.$$

The notation  $\Delta(r, r')$  stands for the least  $n < \omega$  such that  $r(n) \neq r'(n)$  for  $r, r' \in {}^\omega 2$  with  $r \neq r'$ . We also recall.

**Definition.** *BMM* (Bounded Martin's Maximum) holds, if for any  $A \in H_{\omega_2}^V$  and any  $\Sigma_0$ -formula  $\varphi$ , if  $\Vdash_P \exists y \varphi(y, A)$  in  $H_{\omega_2}^{V[G]}$  holds for some preorder  $P$  which preserves every stationary subset of  $\omega_1$ , then we already have  $\exists y \varphi(y, A)$  in  $H_{\omega_2}^V$ .

We may formulate a weaker forcing axiom by restricting the class of preorders to the semiproper ones.

**Definition.** *BSPFA* (Bounded Semi-Proper Forcing Axiom) holds, if for any  $A \in H_{\omega_2}^V$  and any  $\Sigma_0$ -formula  $\varphi$ , if  $\Vdash_P \exists y \varphi(y, A)$  in  $H_{\omega_2}^{V[G]}$  holds for some preorder  $P$  which is semiproper, then we already have  $\exists y \varphi(y, A)$  in  $H_{\omega_2}^V$ .

In [T], it is shown

**Theorem.** ([T]) (1) *BMM* implies  $\theta_{AC}$ ,

(2)  $\theta_{AC}$  implies  $2^\omega = 2^{\omega_1} = \omega_2$ .

In this note, we consider  $\theta_{AC}^*$  which is somewhat stronger than  $\theta_{AC}$  of [T] and show

(3) If BSPFA holds and there exists a measurable cardinal, then  $\theta_{AC}^*$  holds,

(4)  $\theta_{AC}^*$  implies both  $\theta_{AC}$  and CB (Complete Bounding).

While  $\theta_{AC}$  of [T] demands existences of  $\alpha, \beta$  and  $\gamma$  with  $\omega_1 \leq \alpha < \beta < \gamma$ , our  $\theta_{AC}^*$  further demands  $\alpha = \omega_1$ . The consistency strength of the assumption in (1) is not well-known. A proper class of Woodin cardinals suffices (p. 867 in [W]). However they say it is unknown whether BMM implies  $0^\#$  or not.

On the other hand, if we have a type of reflecting cardinal (which itself is very much weaker than Mahlo) and a measurable cardinal above it (and so lots of measurable must exist below it), then we get the consistency of the assumption in (3) via a revised countable support iteration (say, see [M2]).

## § 1. Basics with The One-to-one Lists in The Cantor Space

**1.1 Definitin.** A *one-to-one list*  $\mathbf{r} = \langle r_i \mid i < \omega_1 \rangle$  in  ${}^\omega 2$  means that for all  $i < \omega_1$ ,  $r_i : \omega \rightarrow 2$  and for all  $i, j < \omega_1$ , if  $i \neq j$ , then  $r_i \neq r_j$ . In this case, we denote  $\Delta(r_i, r_j) = \text{Min } \{n < \omega \mid r_i(n) \neq r_j(n)\}$ . More generally, we consider a one-to-one list  $\mathbf{r} = \langle r_i \mid i \in T \rangle$  on a stationary set  $T \subseteq \omega_1$  in  ${}^\omega 2$ . For a countable set  $X$  of ordinals,  $\text{o.t.}(X)$  denotes the order type of  $X$ . Hence  $\text{o.t.}(X) < \omega_1$ . For any ordinals  $\alpha < \beta$ , if  $\text{o.t.}(X \cap \alpha) < \text{o.t.}(X \cap \beta) < \omega_1$ , then we denote  $\Delta_X^r(\alpha, \beta) = \Delta(r_{\text{o.t.}(X \cap \alpha)}, r_{\text{o.t.}(X \cap \beta)})$ . We usually simply write  $\Delta_X(\alpha, \beta)$  instead of  $\Delta_X^r(\alpha, \beta)$ . For any ordinals  $\alpha, \beta$  and  $\gamma$ , if  $\text{o.t.}(X \cap \alpha) < \text{o.t.}(X \cap \beta) < \text{o.t.}(X \cap \gamma) < \omega_1$ , then we denote  $\text{Max } \Delta_X(\alpha, \beta, \gamma) = \text{Max } \{\Delta_X(\alpha, \beta), \Delta_X(\alpha, \gamma), \Delta_X(\beta, \gamma)\}$ .

**1.2 Lemma.** Let  $\mathbf{r} = \langle r_i \mid i < \omega_1 \rangle$  be a one-to-one list in  ${}^\omega 2$ . Then there exists  $n < \omega$  such that both  $\{i < \omega_1 \mid r_i(n) = 0\}$  and  $\{i < \omega_1 \mid r_i(n) = 1\}$  are stationary.

*Proof.* Suppose not. For each  $n < \omega$ , there is a club  $C_n$  and  $\epsilon_n$  such that for all  $i \in C_n$ ,  $r_i(n) = \epsilon_n$ . Let  $C = \bigcap \{C_n \mid n < \omega\}$ . Then  $C$  is a club and for all  $i \in C$  and all  $n < \omega$ , we have  $r_i(n) = \epsilon_n$ . Hence  $\{r_i \mid i \in C\}$  has one element. This is a contradiction.  $\square$

**1.3 Lemma.** Let  $\mathbf{r} = \langle r_i \mid i \in T \rangle$  be a one-to-one list on a stationary set  $T$  in  ${}^\omega 2$ . Then there exist  $m < \omega$  and  $s \in {}^m 2$  such that both  $\{i \in T \mid r_i \restriction m = s \text{ and } r_i(m) = 0\}$  and  $\{i \in T \mid r_i \restriction m = s \text{ and } r_i(m) = 1\}$  are stationary.

*Proof.* Suppose not. For each  $m < \omega$  and  $s \in {}^m 2$ , there exist a club  $C_{ms}$  and  $\epsilon_{ms}$  such that for all  $i \in C_{ms} \cap T$ , we have if  $r_i \restriction m = s$ , then  $r_i(m) = \epsilon_{ms}$ . Let  $C = \bigcap \{C_{ms} \mid m < \omega, s \in {}^m 2\}$ . Then  $C$  is a club and for all  $m < \omega$ , all  $s \in {}^m 2$  and all  $i \in C \cap T$ , we have if  $r_i \restriction m = s$ , then  $r_i(m) = \epsilon_{ms}$ . In particular,  $r_i(m) = \epsilon_{m, r_i \restriction m}$ . Hence for  $i, j \in C \cap T$ , we may show  $r_i \restriction m = r_j \restriction m$  for all  $m < \omega$  by induction on  $m$ . Hence  $\{r_i \mid i \in C \cap T\}$  has one element. This is a contradiction.  $\square$

**1.4 Lemma.** Let  $\mathbf{r} = \langle r_i \mid i < \omega_1 \rangle$  be a one-to-one list in  ${}^\omega 2$ . For any stationary  $S$  and any  $n < \omega$ , there exist  $m$  with  $n \leq m < \omega$  and  $s \in {}^m 2$  such that both  $\{i \in S \mid r_i \restriction m = s \text{ and } r_i(m) = 0\}$  and  $\{i \in S \mid r_i \restriction m = s \text{ and } r_i(m) = 1\}$  are stationary.

*Proof.* Let  $S$  and  $n$  be as given. Since  $\{r_i \restriction n \mid i \in S\}$  is finite,  $S$  gets partitioned into finitely many cells according to  $r_i \restriction n$ . But  $S$  is stationary. Hence one of them is stationary. So there is  $t \in {}^n 2$  such that  $T = \{i \in S \mid r_i \restriction n = t\}$  is stationary. Now may apply lemma 1.3 to a one-to-one list  $\langle r_i \restriction [n, \omega) \mid i \in T \rangle$  (somewhat abusive). Hence there exist  $m$  with  $n \leq m < \omega$  and  $u \in {}^{[n, m)} 2$  such that both  $\{i \in S \mid r_i \restriction n = t, r_i \restriction [n, m) = u, r_i(m) = 0\}$  and  $\{i \in S \mid r_i \restriction n = t, r_i \restriction [n, m) = u, r_i(m) = 1\}$  are stationary.  $\square$

**1.5 Lemma.** Let  $\mathbf{r} = \langle r_i \mid i < \omega_1 \rangle$  be a one-to-one list in  ${}^\omega 2$ . For any  $n < \omega$ , there exists a club  $C_n$  such that for any  $i \in C_n$  there is  $m$  with  $n \leq m < \omega$  such that both  $\{j \in \omega_1 \mid r_j \restriction m = r_i \restriction m, r_j(m) = 0\}$  and  $\{j \in \omega_1 \mid r_j \restriction m = r_i \restriction m, r_j(m) = 1\}$  are stationary.

*Proof.* Suppose not. For any club  $C$ , there is  $i \in C$  such that for any  $m$  with  $n \leq m < \omega$ , there is  $\eta$  such that  $\{j \in \omega_1 \mid r_j \restriction m = r_i \restriction m, r_j(m) = \eta\}$  is not stationary. Let  $S = \{i < \omega_1 \mid \text{for all } m \text{ with } n \leq m < \omega, \text{ there is } \eta \text{ such that } \{j \in \omega_1 \mid r_j \restriction m = r_i \restriction m, r_j(m) = \eta\} \text{ is not stationary}\}$ . Then  $S$  is stationary. By lemma 1.4, we have  $m$  with  $n \leq m < \omega$  and  $s \in {}^m 2$  such that both  $S^0 = \{i \in S \mid r_i \restriction m = s, r_i(m) = 0\}$  and  $S^1 = \{i \in S \mid r_i \restriction m = s, r_i(m) = 1\}$  are stationary. Pick any  $i \in S^0 (\neq \emptyset)$ . Then  $r_i \restriction m = s$  and  $i \in S$ . Hence there is  $\eta$  such that  $\{j \in \omega_1 \mid r_j \restriction m = s, r_j(m) = \eta\}$  is not stationary. Since  $S^0$  is stationary, we have  $\eta = 1$ . Similary, since  $S^1$  is stationary, we have  $\eta = 0$ . This is a contradiction.  $\square$

**1.6 Lemma.** Let  $\mathbf{r} = \langle r_i \mid i < \omega_1 \rangle$  be a one-to-one list in  ${}^\omega 2$ . Then there exists a club  $C_r$  such that for any  $i \in C_r$  and any  $n < \omega$ , there is  $m$  with  $n \leq m < \omega$  such that both  $\{j \in \omega_1 \mid r_j \restriction m = r_i \restriction m, r_j(m) = 0\}$  and  $\{j \in \omega_1 \mid r_j \restriction m = r_i \restriction m, r_j(m) = 1\}$  are stationary.

*Proof.* Let  $C_r = \bigcap \{C_{rn} \mid n < \omega\}$ . Then this  $C_r$  works.  $\square$

**1.7 Lemma.** Let  $r = \langle r_i \mid i < \omega_1 \rangle$  be a one-to-one list in  ${}^\omega 2$ . Then there exists a club  $C_r$  such that for any  $i \in C_r$  and any  $n < \omega$ , we have  $\{j < \omega_1 \mid \Delta(r_i, r_j) \geq n\}$  is stationary.

*Proof.* The  $C_r$  above works.  $\square$

**1.8 Lemma.** Let  $r = \langle r_i \mid i < \omega_1 \rangle$  be a one-to-one list in  ${}^\omega 2$ . Then there exist  $n_r < \omega$  and a club  $C_r$  such that

- Both  $\{j < \omega_1 \mid r_j(n_r) = 0\}$  and  $\{j < \omega_1 \mid r_j(n_r) = 1\}$  are stationary.

And so

- For any  $i < \omega_1$ ,  $\{j < \omega_1 \mid \Delta(r_i, r_j) \leq n_r\}$  is stationary,

While

- For any  $i \in C_r$  and any  $n < \omega$ ,  $\{j < \omega_1 \mid \Delta(r_i, r_j) > n\}$  is stationary.

*Proof.* Let  $n = n_r < \omega$  be any number such that both  $\{j < \omega_1 \mid r_j(n) = 0\}$  and  $\{j < \omega_1 \mid r_j(n) = 1\}$  are stationary. Let  $C_r$  be as in above. These  $n_r$  and  $C_r$  work.  $\square$

## § 2. Basics with Semiproper Preorders

**2.1 Notation.** Let  $\lambda$  be a regular cardinal. We write  $N \prec H_\lambda$ , if the structure  $(N, \in)$  is an elementary substructure of  $(H_\lambda, \in)$ . For  $N$  and  $M$ , we denote  $M \supseteq_{\text{end}} N$ , if  $M \supseteq N$  and  $M \cap \omega_1 = N \cap \omega_1$ . We write  $\langle X_i \mid i < \omega_1 \rangle \nearrow X$ , if  $\langle X_i \mid i < \omega_1 \rangle$  is a sequence of continuously increasing countable subsets of  $X$  and  $\bigcup \{X_i \mid i < \omega_1\} = X$ .

**2.2 Definition.** Let  $\kappa$  be a regular uncountable cardinal and  $S \subseteq [\kappa]^\omega$ . We say  $S$  is *semiproper*, if there exists a club  $C \subseteq [H_{(2^\kappa)^+}]^\omega$  such that for any  $N \prec H_{(2^\kappa)^+}$  with  $N \in C$ , there is a countable  $M \prec H_{(2^\kappa)^+}$  such that  $M \supseteq_{\text{end}} N$  and  $M \cap \kappa \in S$ .

**2.3 Lemma.** Let  $\kappa$  be a regular uncountable cardinal,  $S, T \subseteq [\kappa]^\omega$  be semiproper and disjoint. Then for any  $B \subseteq \omega_1$ , there is a semiproper p.o. set  $P = P(S, T, B)$  such that in  $V^P$ , there is  $\langle X_i \mid i < \omega_1 \rangle \nearrow \kappa$  such that for all  $i < \omega_1$ ,

- If  $i \in B$ , then  $X_i \in S$ ,
- If  $i \notin B$ , then  $X_i \in T$ ,

Hence

- $i \in B$  if and only if  $X_i \in S$ .

*Proof.* Let  $p \in P$ , if  $p = \langle X_i^p \mid i \leq \alpha^p \rangle$  such that

- $p$  is continuously increasing and the  $X_i^p$  are countable subsets of  $\kappa$  with  $\alpha^p < \omega_1$ ,

For  $i \leq \alpha^p$ , we have

- If  $i \in B$ , then  $X_i^p \in S$ ,
- If  $i \notin B$ , then  $X_i^p \in T$ .

For  $p, q \in P$ , let  $q \leq p$ , if  $q \supseteq p$ .

We show that this  $P$  works in a series of claims.

**Claim 1.** For any  $p \in P$  and any  $\xi \in \kappa$ , there is  $X$  such that  $\xi \in X$ ,  $q = p \cup \{(\alpha^p + 1, X)\} \in P$  and  $q \leq p$ .

*Proof.* According to  $\alpha^p + 1 \in B$  or not, we have two cases.

**Case 1.**  $\alpha^p + 1 \in B$ : Since  $S$  is semiproper, there is a countable  $M \prec H_{(2^\kappa)^+}$  such that  $p, \xi \in M$  and  $M \cap \kappa \in S$ . Let  $X = M \cap \kappa$ . Then this  $X$  works.

**Case 2.**  $\alpha^p + 1 \notin B$ : Since  $T$  is semiproper, there is a countable  $M \prec H_{(2^\kappa)^+}$  such that  $p, \xi \in M$  and  $M \cap \kappa \in T$ . Let  $X = M \cap \kappa$ . Then this  $X$  works. □

**Claim 2.** For  $i < \omega_1$  and  $\xi \in \kappa$ ,  $D(i, \xi) = \{q \in P \mid i \leq \alpha^q, \xi \in X_{\alpha^q}^q\}$  is open dense in  $P$ .

*Proof.* By induction on  $i$  for all  $\xi$ . By claim 1, it remains to deal with limit  $i$ . We show this by contradiction. Suppose for any  $q \leq p$ ,  $\alpha^q < i$ . It suffices to derive a contradiction. Let  $\langle i_n \mid n < \omega \rangle$  be increasing such that  $i_0 = \alpha^p$  and  $\sup\{i_n \mid n < \omega\} = i$ . According to  $i \in B$  or not, we have two cases.

**Case 1.**  $i \in B$ : Let  $M \prec H_{(2^\kappa)^+}$  be such that  $i, p, \xi \in M$  and  $M \cap \kappa \in S$ . Let  $\langle \xi_n \mid n < \omega \rangle$  enumerate  $M \cap \kappa$ . By induction we have  $\langle p_n \mid n < \omega \rangle$  so that  $p_0 = p$ ,  $p_n \in P \cap M$ ,  $i_n \leq \alpha^{p_{n+1}} < i$  and  $\xi_n \in X_{\alpha^{p_{n+1}}}^{p_{n+1}}$ . Let  $q = \bigcup\{p_n \mid n < \omega\} \cup \{(i, M \cap \kappa)\}$ . Then  $q \in P$  and  $q \leq p$  with  $\alpha^q = i$ . This is a contradiction.

**Case 2.**  $i \notin B$ : Similarly to case 1, let  $M \prec H_{(2^\kappa)^+}$  be such that  $i, p, \xi \in M$  and  $M \cap \kappa \in T$ . Let  $\langle \xi_n \mid n < \omega \rangle$  enumerate  $M \cap \kappa$ . By induction we have  $\langle p_n \mid n < \omega \rangle$  so that  $p_0 = p$ ,  $p_n \in P \cap M$ ,  $i_n \leq \alpha^{p_{n+1}} < i$  and  $\xi_n \in X_{\alpha^{p_{n+1}}}^{p_{n+1}}$ . Let  $q = \bigcup\{p_n \mid n < \omega\} \cup \{(i, M \cap \kappa)\}$ . Then  $q \in P$  and  $q \leq p$  with  $\alpha^q = i$ . This is a contradiction. □

**Claim 3.**  $P$  is semiproper.

*Proof.* Let  $P \in N \prec H_{(2^\kappa)^+}$  with  $N \in C(S) \cap C(T)$ , where  $C(S)$  and  $C(T)$  are clubs in  $[H_{(2^\kappa)^+}]^\omega$  associated with semiproper  $S$  and  $T$  respectively. Let  $p \in P \cap N$ . We want to find  $q \leq p$  which is  $(P, N)$ -semi-generic. According to  $N \cap \omega_1 \in B$  or not, we have two cases.

**Case 1.**  $N \cap \omega_1 \in B$ : Since  $N \in C(S)$ , we may take a countable  $M \prec H_{(2^\kappa)^+}$  such that  $M \supseteq_{\text{end}} N$  and  $M \cap \kappa \in S$ . Let  $\langle p_n \mid n < \omega \rangle$  be a  $(P, M)$ -generic sequence with  $p_0 = p$ . Let  $q = \bigcup\{p_n \mid n < \omega\} \cup \{(M \cap \omega_1, M \cap \kappa)\}$ . Then by claim 2, we know that  $q \in P$  and so  $q \leq p$ . By construction,  $q$  is  $(P, M)$ -generic and so  $(P, N)$ -semi-generic.

**Case 2.**  $N \cap \omega_1 \notin B$ : Similarly to case 1, take a countable  $M \prec H_{(2^\kappa)^+}$  such that  $M \supseteq_{\text{end}} N$  and  $M \cap \kappa \in T$ . Let  $\langle p_n \mid n < \omega \rangle$  be a  $(P, M)$ -generic sequence with  $p_0 = p$ . Let  $q = \bigcup\{p_n \mid n < \omega\} \cup \{(M \cap \omega_1, M \cap \kappa)\}$ . Then by claim 2, we know that  $q \in P$  and so  $q \leq p$ . By construction,  $q$  is  $(P, M)$ -generic and so  $(P, N)$ -semi-generic. □

**Claim 4.** Let  $G$  be any  $P$ -generic filter over  $V$  and let  $\langle X_i \mid i < \omega_1 \rangle = \bigcup G$ . Then  $\langle X_i \mid i < \omega_1 \rangle \nearrow \kappa$  and for  $i < \omega_1$ , we have

- If  $i \in B$ , then  $X_i \in S$ ,
- If  $i \notin B$ , then  $X_i \in T$ .

*Proof.* By construction of  $P$  and claim 2. Notice that  $|\kappa| = \omega_1$  holds in the extension  $V[G]$ . □

This completes the proof of lemma.

**2.4 Lemma.** Let  $\kappa$  be a regular uncountable cardinal and  $S \subseteq [\kappa]^\omega$  be semiproper. Then there is a semiproper p.o. set  $P = P(S)$  such that in  $V^P$ , there is  $\langle X_i \mid i < \omega_1 \rangle \nearrow \kappa$  such that for all  $i < \omega_1$ ,  $X_i \in S$ .

*Proof.* The proof is entirely similar to and simpler than lemma 2.3. □

### § 3. First Use of A Measurable Cardinal and BSPFA

We prepare a lemma with a measurable cardinal which is by now well-known with stronger statements.

**3.1 Lemma.** Let  $\kappa$  be a measurable cardinal with a normal measure  $D$  on  $\kappa$ . Let  $N$  be a countable elementary substructure of  $H_{(2^\kappa)^+}$  with  $D \in N$ .

- (1) For any  $\eta \in \kappa$  and any  $s \in \bigcap(N \cap D)$  such that  $\sup(N \cap \kappa), \eta < s$ , we may form a countable elementary substructure  $M$  of  $H_{(2^\kappa)^+}$  such that  $N \cup \{s\} \subset M$  and  $M \cap s = N \cap s = N \cap \kappa$ .
- (2) There is a continuously increasing countable elementary substructures  $\langle N_i \mid i < \omega_1 \rangle$  of  $H_{(2^\kappa)^+}$  such that  $N_0 = N$  and  $\langle \text{o.t.}(N_i \cap \kappa) \mid i < \omega_1 \rangle$  is a strictly increasing continuous sequence of countable ordinals.
- (3) For any stationary  $S \subseteq \omega_1$ , there is a countable elementary substructure  $M$  of  $H_{(2^\kappa)^+}$  such that  $N \subseteq_{\text{end}} M$  and  $\text{o.t.}(M \cap \kappa) \in S$ .

*Proof.* For (1): Let  $M = \{f(s) \mid f \in N\}$ . Then this  $M$  works.

For (2): Construct  $\langle N_i \mid i < \omega_1 \rangle$  by recursion on  $i$ . At the successor stages, apply (1). At the limit stages, just take a union.

For (3): Immediate by (2). □

**3.2 Lemma.** Let  $\kappa$  be a measurable cardinal and  $\mathbf{r} = \langle r_i \mid i < \omega_1 \rangle$  be a one-to-one list in  ${}^\omega 2$ . For any countable  $N \prec H_{(2^\kappa)^+}$  with  $\mathbf{r}, \kappa \in N$  and any  $n < \omega$ , there exists a countable  $M \prec H_{(2^\kappa)^+}$  such that  $M \supseteq_{\text{end}} N$  and  $\Delta_M(\omega_1, \kappa) \geq n$ . Namely,  $S(\mathbf{r}, \kappa, n) = \{X \in [\kappa]^\omega \mid \Delta_X(\omega_1, \kappa) \geq n\}$  is semiproper.

*Proof.* Since  $\mathbf{r} \in N$ , we may assume  $C_{\mathbf{r}} \in N$  and so  $\delta = N \cap \omega_1 \in C_{\mathbf{r}}$ . Therefore  $S = \{j < \omega_1 \mid \Delta(r_\delta, r_j) \geq n\}$  is stationary. Since  $\kappa$  is measurable and  $\kappa \in N$ , we may take a countable  $M \prec H_{(2^\kappa)^+}$  such that  $M \supseteq_{\text{end}} N$  and  $j = \text{o.t.}(M \cap \kappa) \in S$ . Hence  $\Delta_M(\omega_1, \kappa) = \Delta(r_{\text{o.t.}(M \cap \omega_1)}, r_{\text{o.t.}(M \cap \kappa)}) = \Delta(r_\delta, r_j) \geq n$ . □

**3.3 Lemma.** Let  $\kappa$  be a measurable cardinal and  $\mathbf{r} = \langle r_i \mid i < \omega_1 \rangle$  be a one-to-one list in  ${}^\omega 2$ . For any  $n < \omega$ , there exists a semiproper p.o. set  $P$  such that in  $V^P$ , there exists  $\langle \dot{X}_i \mid i < \omega_1 \rangle \nearrow \kappa$  such that for all  $i < \omega_1$ ,  $\Delta_{\dot{X}_i}(\omega_1, \kappa) \geq n$ .

*Proof.* Apply lemma 2.4 to  $S(\mathbf{r}, \kappa, n)$ . □

**3.4 Lemma.** (BSPFA) Let a measurable cardinal exist and  $\mathbf{r} = \langle r_i \mid i < \omega_1 \rangle$  be a one-to-one list in  ${}^\omega 2$ . For any  $n < \omega$ , there exists  $\beta$  with  $\omega_1 < \beta < \omega_2$  and  $\langle X_i \mid i < \omega_1 \rangle \nearrow \beta$  such that for all  $i < \omega_1$ ,  $\Delta_{X_i}(\omega_1, \beta) \geq n$ .

*Proof.* Apply BSPFA to lemma 3.3. □

### § 4. Modifications and Summary

**4.1 Lemma.** Let  $n < \omega$ ,  $\omega_1 < \beta < \omega_2$  and  $\langle X_i \mid i < \omega_1 \rangle \nearrow \beta$  be such that for any  $i < \omega_1$ ,  $\Delta_{X_i}(\omega_1, \beta) \geq n$ . Then we have a continuously increasing  $\langle N_i \mid i < \omega_1 \rangle$  such that

- For all  $i < \omega_1$ ,  $N_i \prec H_{\omega_2}$  and  $N_i$  is countable,
- $\beta \in N_0$ ,  $\bigcup \{N_i \mid i < \omega_1\} \supset \omega_1$  and so  $\bigcup \{N_i \mid i < \omega_1\} \supset \beta$ ,
- For all  $i < \omega_1$ ,  $\Delta_{N_i}(\omega_1, \beta) \geq n$ .

*Proof.* Let  $\langle N_i \mid i < \omega_1 \rangle$  be any continuously increasing sequence of countable  $N_i \prec H_{\omega_2}$  such that  $\bigcup \{N_i \mid i < \omega_1\} \supset \omega_1$  and  $\beta \in N_0$ . Then since  $\beta < \omega_2$ , we have  $\bigcup \{N_i \cap \beta \mid i < \omega_1\} = \beta$  and so  $C = \{i < \omega_1 : X_i = N_i \cap \beta\}$  is a club. By reenumerating  $\{N_i \mid i \in C\}$ , we are done.  $\square$

**4.2 Lemma.** (BSPFA) Let a measurable cardinal exist and  $\mathbf{r} = \langle r_i \mid i < \omega_1 \rangle$  be a one-to-one list in  ${}^\omega 2$ . Then there exist  $n_{\mathbf{r}} < \omega$ , a club  $C_{\mathbf{r}}$ ,  $\beta_{\mathbf{r}}$  with  $\omega_1 < \beta_{\mathbf{r}} < \omega_2$  and  $\langle N_i^{\mathbf{r}} \mid i < \omega_1 \rangle$  continuously increasing such that

- Both  $\{j < \omega_1 \mid r_j(n_{\mathbf{r}}) = 0\}$  and  $\{j < \omega_1 \mid r_j(n_{\mathbf{r}}) = 1\}$  are stationary,
- For any  $i < \omega_1$ ,  $\{j < \omega_1 \mid \Delta(r_i, r_j) \leq n_{\mathbf{r}}\}$  is stationary,
- For any  $i \in C_{\mathbf{r}}$  and any  $n < \omega$ ,  $\{j < \omega_1 \mid \Delta(r_i, r_j) > n\}$  is stationary,
- For any  $i < \omega_1$ ,  $N_i^{\mathbf{r}} \prec H_{\omega_2}$  and  $N_i^{\mathbf{r}}$  is countable,
- $\beta_{\mathbf{r}} \in N_0^{\mathbf{r}}$ ,  $\bigcup \{N_i^{\mathbf{r}} \mid i < \omega_1\} \supset \omega_1$  and so  $\bigcup \{N_i^{\mathbf{r}} \cap \beta_{\mathbf{r}} \mid i < \omega_1\} = \beta_{\mathbf{r}}$ ,
- For any  $i < \omega_1$ ,  $\Delta_{N_i^{\mathbf{r}}}(\omega_1, \beta_{\mathbf{r}}) \geq n_{\mathbf{r}} + 1$ .

*Proof.* Combine lemma 1.8, lemma 3.4 and lemma 4.1.  $\square$

## § 5. Second Use of The Same Measurable Cardinal and BSPFA

**5.1 Definition.** Let  $\theta_{AC}^*$  denote the following statement. For any  $\mathbf{r}$  one-to-one list in  ${}^\omega 2$  and any  $B \subseteq \omega_1$ , there exist  $\beta$  and  $\gamma$  with  $\omega_1 < \beta < \gamma < \omega_2$  and  $\langle X_i \mid i < \omega_1 \rangle \nearrow \gamma$  such that for any  $i < \omega_1$ ,  $i \in B$  if and only if  $\Delta_{X_i}(\omega_1, \beta) = \text{Max } \Delta_{X_i}(\omega_1, \beta, \gamma)$ .

It is clear that  $\theta_{AC}^*$  implies  $\theta_{AC}$  of [T].

**5.2 Theorem.** (BSPFA) If there exists a measurable cardinal, then  $\theta_{AC}^*$  holds.

We show this in a series of lemmas.

**5.3 Lemma.** Let  $\kappa$  be a measurable cardinal and  $\mathbf{r}$  be a one-to-one list in  ${}^\omega 2$ . For any  $\beta$  with  $\omega_1 < \beta < \kappa$ , any countable  $N \prec H_{(2^\kappa)^+}$  with  $\mathbf{r}, \beta, \kappa \in N$ , there exists a countable  $M \prec H_{(2^\kappa)^+}$  such that  $M \supseteq_{\text{end}} N$  and  $\Delta_M(\omega_1, \beta) = \text{Min } \Delta_M(\omega_1, \beta, \kappa)$ .

*Proof.* Since  $\mathbf{r} \in N$ , we may assume  $C_{\mathbf{r}} \in N$  and so  $N \cap \omega_1 \in C_{\mathbf{r}}$ . Hence for all  $n < \omega$ , we have  $\{j < \omega_1 \mid \Delta(r_{N \cap \omega_1}, r_j) \geq n\}$  is stationary. Since  $\omega_1 < \beta$ , we may calculate  $\Delta_N(\omega_1, \beta) = n$ . Since  $\kappa$  is a measurable cardinal, we may choose a countable  $M \prec H_{(2^\kappa)^+}$  such that  $M \cap \beta = N \cap \beta$ , if  $j = \text{o.t.}(M \cap \kappa)$ , then  $\Delta(r_{N \cap \omega_1}, r_j) \geq n + 1$ . Since  $\Delta_M(\omega_1, \beta) = \Delta_N(\omega_1, \beta) = n < \Delta(r_{N \cap \omega_1}, r_{\text{o.t.}(M \cap \kappa)}) = \Delta_M(\omega_1, \kappa)$ , we have  $\Delta_M(\omega_1, \beta) = \text{Min } \Delta_M(\omega_1, \beta, \kappa)$ .  $\square$

**5.4 Lemma.** (BSPFA) Let  $\kappa$  be a measurable cardinal and  $\mathbf{r}$  be a one-to-one list in  ${}^\omega 2$ . For any countable  $N \prec H_{(2^\kappa)^+}$  with  $\mathbf{r}, \kappa \in N$ , there exists a countable  $M \prec H_{(2^\kappa)^+}$  such that  $M \supseteq_{\text{end}} N$  and  $\Delta_M(\omega_1, \beta_{\mathbf{r}}) = \text{Max } \Delta_M(\omega_1, \beta_{\mathbf{r}}, \kappa)$ .

*Proof.* Let  $\eta = r_{N \cap \omega_1}(n_r)$ . Let  $\bar{\eta} \in \{0, 1\}$  and  $\eta \neq \bar{\eta}$ . Since  $\{j < \omega_1 \mid r_j(n_r) = \bar{\eta}\}$  is stationary, we may choose a countable  $M \prec H_{(2^\kappa)^+}$  such that  $M \supseteq_{\text{end}} N$  and  $r_{\text{o.t.}(M \cap \kappa)}(n_r) = \bar{\eta}$ . Hence  $\Delta_M(\omega_1, \kappa) \leq n_r$ . On the other hand, since we may assume  $\langle N_i^r \mid i < \omega_1 \rangle \in N$ , if  $\delta = N \cap \omega_1$ , then we have  $N_\delta^r \subseteq_{\text{end}} N \cap H_{\omega_2}$ . Since  $\beta_r \in N_0^r$ , we conclude  $N_\delta^r \cap \beta_r = N \cap \beta_r = M \cap \beta_r$  holds. So  $\Delta_M(\omega_1, \beta_r) = \Delta_N(\omega_1, \beta_r) = \Delta_{N_\delta^r}(\omega_1, \beta_r) \geq n_r + 1$ . Therefore,  $\Delta_M(\omega_1, \beta_r) = \text{Max } \Delta_M(\omega_1, \beta_r, \kappa)$ .  $\square$

**5.5 Lemma.** (BSPFA) Let  $\kappa$  be a measurable cardinal and  $\mathbf{r}$  be a one-to-one list in  ${}^\omega 2$ . Let  $S = \{X \in [\kappa]^\omega \mid \Delta_X(\omega_1, \beta_r) = \text{Max } \Delta_X(\omega_1, \beta_r, \kappa)\}$  and  $T = \{X \in [\kappa]^\omega \mid \Delta_X(\omega_1, \beta_r) = \text{Min } \Delta_X(\omega_1, \beta_r, \kappa)\}$ . Then both  $S$  and  $T$  are semiproper and disjoint.

*Proof.* By lemma 5.4 and lemma 5.3.  $\square$

**5.6 Lemma.** (BSPFA) Let  $\kappa$  be a measurable cardinal. Let  $\mathbf{r} = \langle r_i \mid i < \omega_1 \rangle$  be a one-to-one list in  ${}^\omega 2$  and  $B \subseteq \omega_1$ . Then there exists a semiproper p.o. set  $P$  such that in  $V^P$ , there is  $\langle Y_i \mid i < \omega_1 \rangle \nearrow \kappa$  such that for any  $i < \omega_1$ ,  $i \in B$  if and only if  $\Delta_{Y_i}(\omega_1, \beta_r) = \text{Max } \Delta_{Y_i}(\omega_1, \beta_r, \kappa)$ .

*Proof.* By lemma 5.5 and lemma 2.3.  $\square$

*Proof of theorem 5.2.* Apply BSPFA to the p.o. set in lemma 5.6.  $\square$

## § 6. $\theta_{\text{AC}}^*$ implies CB

**6.1 Definition.** CB (complete bounding) stands for the following. For any  $f : \omega_1 \rightarrow \omega_1$ , there exist  $\omega_1 < \gamma < \omega_2$ , a club  $C$  and  $\langle X_i \mid i < \omega_1 \rangle \nearrow \gamma$  such that for all  $i \in C$ ,  $f(i) < \text{o.t.}(X_i)$ .

**6.2 Theorem.**  $\theta_{\text{AC}}^*$  implies CB.

*Proof.* We have two claims.

**Claim 1.** If for any one-to-one list  $\mathbf{r}$  in  ${}^\omega 2$ , there exist  $\omega_1 < \beta < \omega_2$  and  $\langle X_i \mid i < \omega_1 \rangle \nearrow \beta$  such that  $\Delta_{X_i}(\omega_1, \beta) > 0$ , then CB holds.

*Proof.* Let  $f : \omega_1 \rightarrow \omega_1$ . We may assume that for all  $i < \omega_1$ ,  $i < f(i)$  and  $f$  is strictly increasing. Take a continuously increasing  $\langle N_i \mid i < \omega_1 \rangle$  such that each  $N_i$  is countable,  $N_i \prec H_{\omega_2}$  and  $N_i \in N_{i+1}$  and  $f \in N_0$ . Notice that  $N_i \cap \omega_1 < f(N_i \cap \omega_1) < N_{i+1} \cap \omega_1$ . It is easy to construct  $\mathbf{r}$  so that  $r_{N_i \cap \omega_1}(0) = 1$ , for  $\xi$  with  $N_i \cap \omega_1 < \xi \leq f(N_i \cap \omega_1)$ , we have  $r_\xi(0) = 0$ . By assumption get  $\omega_1 < \beta < \omega_2$  and  $\langle X_i \mid i < \omega_1 \rangle$  such that for all  $i < \omega_1$ , we have  $\Delta_{X_i}(\omega_1, \beta) > 0$ . Let  $C = \{i < \omega_1 \mid N_i \cap \omega_1 = i = X_i \cap \omega_1, \omega_1 \in X_i\}$ . Then for  $i \in C$ , since  $\Delta_{X_i}(\omega_1, \beta) = \Delta(r_i, r_{\text{o.t.}(X_i)}) > 0$ , we have  $f(i) = f(N_i \cap \omega_1) < \text{o.t.}(X_i)$ .  $\square$

**Claim 2.** If  $\theta_{\text{AC}}^*$  holds, then for any one-to-one list  $\mathbf{r}$  in  ${}^\omega 2$ , there exist  $\omega_1 < \beta < \omega_2$  and  $\langle X_i \mid i < \omega_1 \rangle \nearrow \beta$  such that  $\Delta_{X_i}(\omega_1, \beta) > 0$ .

*Proof.* By  $\theta_{\text{AC}}^*$  for  $B = \omega_1$ , there exist  $\omega_1 < \beta < \gamma < \omega_2$  and  $\langle Y_i \mid i < \omega_1 \rangle \nearrow \gamma$  such that for all  $i < \omega_1$ , we have  $\Delta_{Y_i}(\omega_1, \beta) = \text{Max } \Delta_{Y_i}(\omega_1, \beta, \gamma)$ . In particular,  $\Delta_{Y_i}(\omega_1, \beta) > 0$ . Let  $X_i = Y_i \cap \beta$ . Then these  $\beta$  and  $\langle X_i \mid i < \omega_1 \rangle$  work.



## § 7. Additional Observations

Now we make a few observations. We may consider to directly force our  $\theta_{AC}^*$ . Namely, we may add the following to [D].

**Theorem.** ([D], [M]) The following are equiconsistent.

- $\text{Con}(\text{There exists a regular cardinal } \rho \text{ such that } \{\kappa < \rho \mid \kappa \text{ is a measurable cardinal}\} \text{ is cofinal in } \rho)$ .
- $\text{Con}(\theta_{AC}^*)$ ,
- $\text{Con}(\text{CB})$ .

□

Hence  $\theta_{AC}$  of [T] accordingly has a large cardinal upper-bound.

Next, similarly to  $\text{Con}(\text{PFA}^+ + \neg \text{CB})$  (which we got from S. Todorcevic), we may show via  $\omega_1$ -many Cohen reals  $\mathbf{r}$ ,

**Theorem.** ([M])  $\text{Con}(\text{PFA}^+ \text{ and } \neg \theta_{AC})$  and so  $\text{Con}(\text{PFA}^+ \text{ and } \neg \text{BMM})$ .

□

Lastly, starting with a Souslin tree in the ground model and preserving it ([M1]), we have

**Theorem.** ([M])  $\text{Con}(\text{There exists a Souslin tree and } \theta_{AC}^*)$  and so  $\text{Con}(\neg \text{MA and } \theta_{AC}^*)$

□

Among others concerning the large cardinal strength of BMM, we may ask

**Question.** Does  $\theta_{AC}$  of [T] imply any large cardinal, say, CB ?

More modestly,

**Question.** Does BMM imply the Weak Chang's Conjecture ?

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